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On the super-potentials for Liénard–Wiechert potentials in far fields

Hideki Kawaguchi and Toshihisa Honma

Department of Electrical Engineering, Faculy of Engineering, Hokkaido University, Kita 13, Nishi 8, Kita-ku, Sapporo, Japan

Abstract. There exist super-potentials which result in Liénard-Wiechert potentials. The physical meaning of super-potentials is the coordinate of the particle which produces the Liénard-Wiechert potentials. In this paper, the super-potentials in far fields are considered analytically and numerically. It is shown that the super-potentials can be calculated from the far electric fields produced by the particle, if the motion is periodic and non-relativistic. This means that one can estimate the particle trajectory from the far electric fields.

1. Introduction

Liénard-Wiechert potentials are the most exact potentials in classical electrodynamics, because they are solutions to an inhomogeneous wave equation which describes electromagnetic fields produced by a moving point charged particle. These potentials have been studied by many researchers and various electrodynamic phenomena have been predicted using them. One of the examples is synchrotron radiation [1, 2]. However, there is a mysterious formula relevant to Liénard–Wiechert potentials. The formula tells us that there exist super-potentials which result in Liénard–Wiechert potentials [3]. On the other hand, the meaning of the formula remains unclear, even now, though Liénard-Wiechert potentials were introduced in 1898.

In this paper, the super-potentials in far fields will be considered analytically and numerically. It will be shown that the super-potentials can be calculated from the far electric fields produced by the particle using the mysterious formula, when the motion is periodic and non-relativistic. On the other hand, the physical meaning of super-potentials is the coordinate of the particle. It means that one can estimate a charged particle trajectory using the far electric fields.

2. Some formulae relevant to Liénard–Wiechert poentials

In this section, the standard representation of Liénard-Wiechert potentials is summarized for later reference. And then, super-potentials for Liénard-Wiechert potentials are introduced and some formulas relevant to the super-potentials are presented.

Liénard–Wiechert potentials A^i are solutions to inhomogeneous wave equation which describes electromagnetic fields produced by a moving point charged particle [4].

$$\Box A^i(ct, \boldsymbol{x}) = \frac{e}{\varepsilon_0 c^2} u^i(t)\delta[\boldsymbol{x} - \boldsymbol{y}(t)] \tag{1}$$

where $\Box$ is the D'Alembertian, $u^i = (c, \boldsymbol{v}(t))$ is the four-velocity of the particle, $\boldsymbol{y}(t)$ is the trajectory of the particle with a parameter t, e is the elementary charge and ε_0 is a dielectric constant. Then Liénard–Wiechert potentials are written as follows:

$$A^i(ct, \boldsymbol{x}) = \frac{e}{4\pi\varepsilon_0 c^2} \frac{u^i(\tau)}{R_k(\tau) u^k(\tau)} \tag{2}$$

where c is the velocity of light, $R_k(\tau)$ is the displacement vector defined by $R_k(\tau) = x^i - y^i(\tau)(y^i(\tau)$ is the four-dimensional position vector of the particle defined by $(c\tau, \boldsymbol{y}(\tau)))$. And then, τ is the so-called 'retarded time' which satisfies a following causal relation:

$$\tau = t - \frac{|\boldsymbol{x} - \boldsymbol{y}(\tau)|}{c}. \tag{3}$$

Liénard–Wiechert potentials depend on the time t and the position $\boldsymbol{x}$ through this recursive relation.

Here the equation (3) can be rewritten in the following form

$$\tau = \tau(ct, \boldsymbol{x}) \tag{4}$$

because retarded time τ is determined uniquely for any t and $\boldsymbol{x}$ [4]. Then one can regard the four-dimensional position vector $y^i(\tau)$ as a function of ct and $\boldsymbol{x}$. That is to say,

$$y^i(\tau) = y^i[\tau(ct, \boldsymbol{x})] = y^i(ct, \boldsymbol{x}). \tag{5}$$

These functions are just super-potentials for Liénard–Wiechert potentials, because Liénard–Wiechert potentials can be expressed using these functions $y^i(\tau)$ as follows [3]:

$$A^i(ct, \boldsymbol{x}) = -\frac{e}{8\pi\varepsilon_0 c} \Box y^i[\tau(ct, \boldsymbol{x})]. \tag{6}$$

Therefore, the super-potentials $y^i(\tau)$ satisfy the following identical equation:

$$\Box^2 y^i[\tau(ct, \boldsymbol{x})] = -\frac{8\pi}{c} u^i(t) \delta[\boldsymbol{x} - \boldsymbol{y}(t)]. \tag{7}$$

Equation (6) is mysterious because the equation tells us that *coordinates of the electromagnetic system* $A^i(ct, \boldsymbol{x})$ are related to *coordinates of the particle* $y^i(\tau)$ by the D'Alembertian $\Box$, directly.

3. A relation between a charged particle trajectory and the far fields

In this section, a relation between a charged particle trajectory (or super-potentials) and the far electric fields produced by the particle is presented.

One can also derive the equation (6) from the Fourier expansion of Liénard–Wiechert potentials $A^i(ct, \boldsymbol{x})$ and super-potentials $y^i(\tau)$. If the motion of a particle is periodic, Fourier expansion of $y^i(\tau)$ is written as follows (see the appendix):

$$y^i(ct, \boldsymbol{x}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{C_n}{n} \int_a^b \sin\left\{n\left[\omega t - \sigma - \frac{\omega}{c} R(\boldsymbol{x}; \sigma)\right]\right\} \mathrm{d}y^i(\sigma) \tag{8}$$

where ω is an angular frequency. Then $\Box y^i(\tau)$ becomes

$$\begin{aligned}\Box y^i(ct, \boldsymbol{x}) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{C_n}{n} \int_a^b \Box \sin\left\{n\left[\omega t - \sigma - \frac{\omega}{c} R(\boldsymbol{x}; \sigma)\right]\right\} \mathrm{d}y^i(\sigma) \\ &= -\frac{\omega}{2\pi c} \sum_{n=0}^{\infty} C_n \int_a^b \frac{\cos\{n[\omega t - \sigma - (\omega/c) R(\boldsymbol{x}; \sigma)]\}}{R(\sigma)} \mathrm{d}y^i(\sigma). \end{aligned} \tag{9}$$

On the other hand, Fourier expansion of Liénard–Wiechert potentials is [3]

$$A^i(ct, \boldsymbol{x}) = \frac{e}{8\pi\varepsilon_0 c^2} \sum_{n=0}^{\infty} C_n \int_a^b \frac{\cos\{n[\omega t - \sigma - (\omega/c)R(\boldsymbol{x}; \sigma)]\}}{R(\sigma)} \, \mathrm{d}y^i(\sigma). \tag{10}$$

Comparing equation (9) with equation (10), it is found that equation (9) is just Fourier expansion of Liénard–Wiechert potentials except for the factor $-e/8\pi\varepsilon_0 c$. Thus, the equation (6) was proved using the Fourier expansion of $A^i(ct, \boldsymbol{x})$ and $y^i(\tau)$.

From equations (6), (8) and (10), spatial components of $y^i(\tau)$ and $A^i(ct, \boldsymbol{x})$ are as follows:

$$\boldsymbol{y}(ct, \boldsymbol{x}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{C_n}{n} \int_a^b \sin\{n[\omega t - \sigma - (\omega/c)R(\boldsymbol{x}; \sigma)]\} \, \mathrm{d}\boldsymbol{y}(\sigma) \tag{11}$$

$$\boldsymbol{A}(ct, \boldsymbol{x}) = -\frac{ew}{16\pi^2\varepsilon_0 c^2} \sum_{n=0}^{\infty} C_n \int_a^b \frac{\cos\{n[\omega t - \sigma - (\omega/c)R(\boldsymbol{x}; \sigma)]\}}{R(\sigma)} \, \mathrm{d}\boldsymbol{y}(\sigma). \tag{12}$$

Moreover, $\boldsymbol{y}(ct, \boldsymbol{x})$ can be also expressed by

$$\boldsymbol{y}(ct, \boldsymbol{x}) = \boldsymbol{y}_0(ct, \boldsymbol{x}) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \sin\{n[\omega t - \sigma - (\omega/c)R(\boldsymbol{x}; \sigma)]\} \, \mathrm{d}\boldsymbol{y}(\sigma) \tag{13}$$

where $\boldsymbol{y}_0(ct, \boldsymbol{x})$ is the term for $n = 0$ in the equation (11). Taking note that $\mathrm{d}\boldsymbol{y}_0/\omega \, \mathrm{d}t = \boldsymbol{0}$ in equation (A7), $\mathrm{d}\boldsymbol{y}_0$ has to depend on $\boldsymbol{x}$ only. It can be found from numerical calculation that $\boldsymbol{y}_0(\boldsymbol{x})$ is expressed as follows:

$$\boldsymbol{y}_0(\boldsymbol{x}) = -\frac{1}{\pi} \frac{\omega}{c} \int_a^b R(\boldsymbol{x}; \sigma) \, \mathrm{d}\boldsymbol{y}(\sigma). \tag{14}$$

For far fields (i.e. $R(\boldsymbol{x}; \sigma) \simeq R(\boldsymbol{x}) - \boldsymbol{x}\cdot\boldsymbol{y}(\sigma)/R(\boldsymbol{x})$)

$$\begin{aligned} \boldsymbol{y}_0(\boldsymbol{x}) &\simeq \frac{1}{2\pi} \frac{\omega}{c} \frac{1}{R(\boldsymbol{x})} \int_a^b [\boldsymbol{x}\cdot\boldsymbol{y}(\sigma)] \, \mathrm{d}\boldsymbol{y}(\sigma) \\ &= -\frac{1}{4\pi} \frac{\omega}{c} \frac{\boldsymbol{x}}{R(\boldsymbol{x})} \int_a^b \boldsymbol{y}(\sigma) \, \mathrm{d}\boldsymbol{y}(\sigma) \\ &= \boldsymbol{M} \times \frac{\boldsymbol{x}}{R(\boldsymbol{x})} \end{aligned} \tag{15}$$

where $R(\boldsymbol{x})$ is the distance from the centre of a charged particle trajectory to observation point, the vector $\boldsymbol{M}$ is a constant defined by

$$\boldsymbol{M} \equiv \frac{1}{4\pi} \frac{\omega}{c} \int_a^b \boldsymbol{y}(\sigma) \times \mathrm{d}\boldsymbol{y}(\sigma). \tag{16}$$

Here, taking note that far electric fields are calculated by

$$\begin{aligned} \boldsymbol{E}(ct, \boldsymbol{x}) &= -\frac{\partial}{\partial t} \boldsymbol{A}(ct, \boldsymbol{x}) \\ &= -\frac{e\omega^2}{8\pi^2\varepsilon_0 c^2} \sum_{n=1}^{\infty} n \int_a^b \frac{\sin\{n[\omega t - \sigma - (\omega/c)R(\boldsymbol{x}; \sigma)]\}}{R(\boldsymbol{x}; \sigma)} \, \mathrm{d}\boldsymbol{y}(\sigma) \end{aligned} \tag{17}$$

one can find a following relation using equations (13), (15) and (17):

$$\boldsymbol{y}(ct, \boldsymbol{x}) \simeq \boldsymbol{M} \times \frac{\boldsymbol{x}}{R(\boldsymbol{x})} - \frac{8\pi\varepsilon_0 c^2}{e\omega^2} R(\boldsymbol{x}) \sum_{n=1}^{\infty} \frac{\boldsymbol{E}_n(ct, \boldsymbol{x})}{n^2}. \tag{18}$$

Equation (18) is just the relation between a charged particle trajectory (or super-potentials) and the far electric fields produced by the particle. It is necessary to evaluate the first term of equation (18) to calculate the particle trajectory $\boldsymbol{y}(ct, \boldsymbol{x})$ from the far fields $\boldsymbol{E}_n(ct, \boldsymbol{x})$. It is clear that the first term of equation (18) can be neglected, when the motion is non-relativistic, because taking note that $|\boldsymbol{x}/R(\boldsymbol{x})| \leqslant 1$ and $|(\omega/c)\boldsymbol{y}(\sigma)| \sim v/c$ (v is the veloctiy of the particle), one can find the following relation:

$$\begin{aligned} \boldsymbol{y}_0(\boldsymbol{x}) &\simeq \frac{1}{2\pi}\frac{\omega}{c}\frac{1}{R(\boldsymbol{x})}\int_a^b [\boldsymbol{x}\cdot\boldsymbol{y}(\sigma)]\, \mathrm{d}\boldsymbol{y}(\sigma) \\ &\leqslant \frac{1}{2\pi}\frac{v}{c}\int_a^b \mathrm{d}\boldsymbol{y}(\sigma) \\ &\ll \text{the second term of equation (13).} \end{aligned} \tag{19}$$

Now, the value Ψ defined by

$$\Psi \equiv -\frac{8\pi\varepsilon_0 c^2}{e\omega^2}\sum_{n=1}^{\infty}\frac{\boldsymbol{E}_n(ct, \boldsymbol{x})}{n^2} \tag{20}$$

represents the trajectory shape divided by distance $R(\boldsymbol{x})$, when the motion is non-relativistic. If far electric fields are observed at some points, one can calculate their

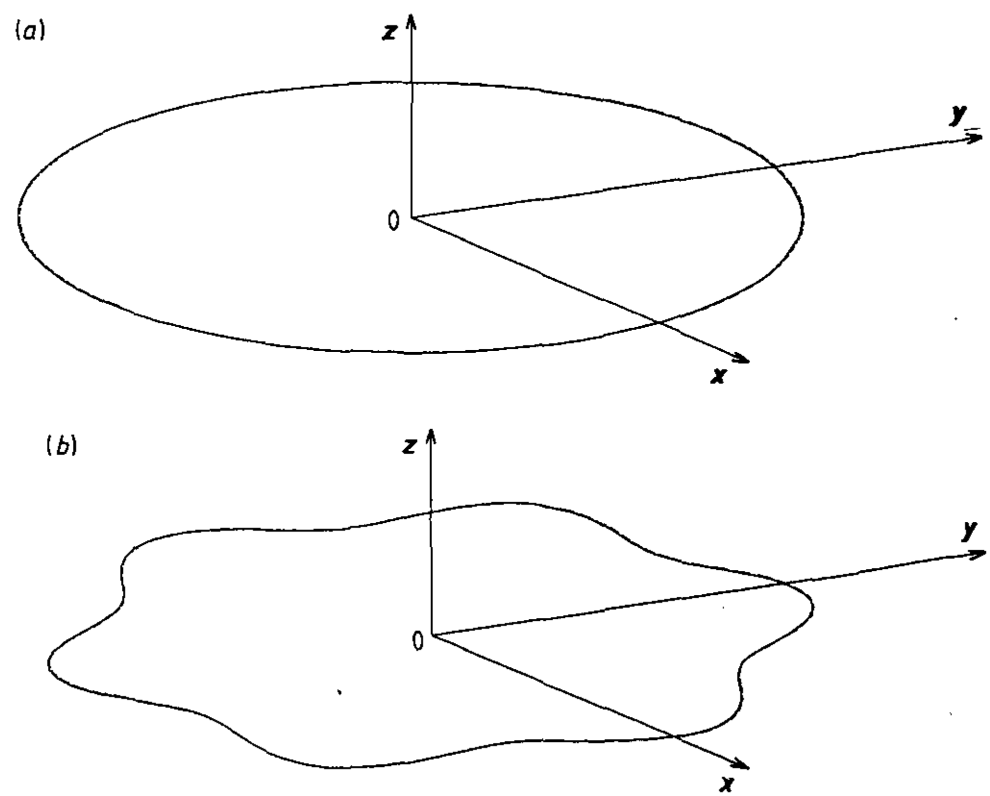

Figure 1. Comparisons of true trajectory (broken line) with trajectory (full line) calculated using equation (18) for (*a*) circular motion and (*b*) betatron motion. $\beta = v/c = 0.01$.

trajectory shapes $\Psi_1, \Psi_2, \ldots$ using equation (20). Differences between these trajectory shapes are only their amplitudes which are $1/R_1(x), 1/R_2(x), \ldots$. Therefore calculating values Ψ_1, Ψ_2, Ψ_3, Ψ_4 for any four points and comparing their amplitudes, the centre of trajectory can be estimated. Consequently, one can say that a charged particle trajectory $y(ct, x)$ is calculated by the far electric fields, if the motion is periodic and non-relativistic. Two examples (circular and betatron motion) are shown in figures 1(*a*) and (*b*). A broken line denotes a true trajectory and a full line denotes the trajectory calculated by an equation (18) in each figure. One can find a good agreement.

4. Summary

In this paper, a relation between a charged particle trajectory and the far electric fields produced by the particle was presented using the super-potentials for Liénard–Wiechert potentials.

The coordinates of the electromagnetic system $A^i(ct, x)$ are related with those coordinates of the particle $y^i(\tau)$ by equation (6). And then equation (18) was derived considering the equation (6) in far fields. This is a reason why one can estimate a particle trajectory from the far fields.

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Appendix

Whittaker's procedure is used [6]. Equation (3) is also written as

$$\omega t = \omega\tau + \omega\frac{|x - y(\omega\tau)|}{c} \equiv \omega\tau + \frac{\omega}{c}R(x; \omega\tau). \tag{A1}$$

Differentiating equation (A1)

$$\omega\, dt = \omega\, d\tau + \frac{\omega}{c}\, dR(x; \omega\tau). \tag{A2}$$

Dividing equation (A2) by differentiation of $y^i(\omega\tau)(dy^i(\omega\tau))$ and expanding its inversion in Fourier series we obtain

$$\frac{\omega\, dt}{dy^i(\omega\tau)} = \frac{\omega\, d\tau}{dy^i(\omega\tau)} + \frac{\omega\, dR(x; \omega\tau)}{c\, dy^i(\omega\tau)}$$

$$\begin{aligned}\frac{dy^i(\omega\tau)}{\omega\, dt} &= \left(\frac{\omega\, d\tau}{dy^i(\omega\tau)} + \frac{\omega\, dR(x; \omega\tau)}{c\, dy^i(\omega\tau)}\right)^{-1} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} du \exp(-inu) \\ &\quad \times \exp(in\omega t)\left(\frac{\omega\, d\tau}{dy^i(\omega\tau)} + \frac{\omega\, dR(x; \omega\tau)}{c\, dy^i(\omega\tau)}\right)^{-1}.\end{aligned} \tag{A3}$$

Transforming the variable u into σ defined by

$$u=\sigma+\frac{\omega}{c}R(x;\sigma) \tag{A4}$$

we obtain

$$\frac{\mathrm{d}y^i(\omega\tau)}{\omega\,\mathrm{d}t}=\frac{1}{2\pi}\sum_{n=0}^{\infty}C_n\int_a^b\cos\left\{n\left[\omega t-\sigma-\frac{\omega}{c}R(x;\sigma)\right]\right\}\mathrm{d}y^i(\sigma) \tag{A5}$$

where $C_n=1$ for $n=0$, $C_n=2$ for $n\neq 0$ and b $(=a+2\pi)$ is determined by the following relation

$$2\pi=b+\frac{\omega}{c}R(x;b). \tag{A6}$$

And then integrating equation (A5) with respect to ωt, equation (8) is derived. Now, it should be noticed that the term for $n=0$ in equation (A5) $(\mathrm{d}y_0^i(\omega\tau/\omega t)$ is written as follows:

$$\begin{aligned}\frac{\mathrm{d}y_0^i(\omega\tau)}{\omega\,\mathrm{d}t}&=\frac{1}{2\pi}\int_a^b\mathrm{d}y^i(\sigma)\\&=(1,0).\end{aligned} \tag{A7}$$

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